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Relativistic quark-antiquark potential and heavy quarkonium mass spectra

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Abstract: A general approach to accounting for retardation effects in the long-range (confining) part of the quark-antiquark potential is presented. The charmonium and bottomonium mass spectra are calculated with the systematic account of relativistic and retardation effects and the one-loop radiative corrections. A good fit to available experimental data on the mass spectra is obtained.

The relativistic properties of the quark-antiquark interaction potential play an important role in analysing different static and dynamical characteristics of heavy mesons. The Lorentz-structure of the confining quark-antiquark interaction is of particular interest. In the literature there is no consent on this item. For a long time the scalar confining kernel has been considered to be the most appropriate one [1]. The main argument in favour of this choice is based on the nature of the heavy quark spin-orbit potential. The scalar potential gives a vanishing long-range magnetic interaction, which is in agreement with the flux tube picture of quark confinement of [2], and allows to get the fine structure for heavy quarkonia in accord with experimental data. However, the calculations of electroweak decay rates of heavy mesons with a scalar confining potential alone yield results which are in worse agreement with data than with a vector potential [3]. The radiative $M1$-transitions in quarkonia such as e. g. $J/\psi \to \eta_c\gamma$ are the most sensitive to the Lorentz-structure of the confining potential. The relativistic corrections for these decays arising from vector and scalar potentials have different signs [3]. In particular, as it has been shown in ref. [3], agreement with experiments for these decays can be achieved only for a specific mixture of vector and scalar potentials. In this context, it is worth noting, that the recent study of the $q\bar{q}$ interaction in the Wilson loop approach [4] indicates that it cannot be considered as purely scalar. Moreover, the found structure of spin-independent relativistic corrections is not compatible with a scalar potential. A similar conclusion has been obtained in ref. [5] on the basis of a Foldy-Wouthuysen reduction of the full Coulomb gauge Hamiltonian of QCD. There, the Lorentz-structure of the confinement has been found to be of vector nature. The scalar nature of spin splittings in heavy quarkonia in this approach is dynamically generated through the interaction with collective gluonic degrees of freedom. Thus we see that while the spin-dependent structure of $q\bar{q}$ interaction is well established now, the spin-independent part is still controversial in the literature. The uncertainty in the Lorentz-structure of the confining interaction complicates the account of retardation corrections since the relativistic reconstruction of the static confining potential is not unique. Here we present the generalized prescription of such reconstruction and discuss its implications for the heavy quarkonium mass spectra.

In our preceding papers we have developed the relativistic quark model based on the quasipotential approach. A meson is described by the wave function of the bound quark-antiquark state, which satisfies the quasipotential equation [6] of the Schrödinger type [7]

$$\left(\frac{b^2(M)}{2\mu_R}-\frac{\mathbf{p}^2}{2\mu_R}\right)\Psi_M(\mathbf{p}) = \int\frac{d^3q}{(2\pi)^3}V(\mathbf{p},\mathbf{q};M)\Psi_M(\mathbf{q}), \qquad (1)$$

where the relativistic reduced mass is

$$\mu_R=\frac{E_aE_b}{E_a+E_b}=\frac{M^4-(m_a^2-m_b^2)^2}{4M^3}, \qquad (2)$$

and E_a, E_b are given by

$$E_a=\frac{M^2-m_b^2+m_a^2}{2M},\quad E_b=\frac{M^2-m_a^2+m_b^2}{2M}. \qquad (3)$$

Here $M=E_a+E_b$ is the meson mass, $m_{a,b}$ are the masses of light and heavy quarks, and $\mathbf{p}$ is their relative momentum. In the centre of mass system the relative momentum squared on mass shell reads

$$b^2(M)=\frac{[M^2-(m_a+m_b)^2][M^2-(m_a-m_b)^2]}{4M^2}. \qquad (4)$$

The kernel $V(\mathbf{p},\mathbf{q};M)$ in Eq. (1) is the quasipotential operator of the quark-antiquark interaction. It is constructed with the help of the off-mass-shell scattering amplitude, projected onto the positive energy states. Constructing the quasipotential of the quark-antiquark interaction we have assumed that the effective interaction is the sum of the usual one-gluon exchange term with the mixture of long-range vector and scalar linear confining potentials, where the vector confining potential contains the Pauli interaction. The quasipotential is then defined by [8]

$$V(\mathbf{p},\mathbf{q};M)=\bar u_a(p)\bar u_b(-p)\Big\{\frac{4}{3}\alpha_sD_{\mu\nu}(\mathbf{k})\gamma_a^\mu\gamma_b^\nu + V_V(\mathbf{k})\Gamma_a^\mu\Gamma_{b;\mu}+V_S(\mathbf{k})\Big\}u_a(q)u_b(-q), \qquad (5)$$

where α_S is the QCD coupling constant, $D_{\mu\nu}$ is the gluon propagator in the Coulomb gauge and $\mathbf{k}=\mathbf{p}-\mathbf{q}$; γ_μ and $u(p)$ are the Dirac matrices and spinors with $\epsilon(p)=\sqrt{p^2+m^2}$. The effective long-range vector vertex is given by

$$\Gamma_\mu(\mathbf{k})=\gamma_\mu+\frac{i\kappa}{2m}\sigma_{\mu\nu}k^\nu, \qquad (6)$$

where κ is the Pauli interaction constant characterizing the anomalous chromomagnetic moment of quarks. Vector and scalar confining potentials in the nonrelativistic limit reduce to

$$V_V(r)=(1-\varepsilon)Ar+B,\quad V_S(r)=\varepsilon Ar, \qquad (7)$$

reproducing

$$V_{\rm conf}(r)=V_S(r)+V_V(r)=Ar+B, \qquad (8)$$

where ε is the mixing coefficient.

The retardation contribution to the one-gluon exchange part of the $q\bar q$ potential is well known. For the confining part of the $q\bar q$ potential the retardation contribution is much more indefinite. It is a consequence of our poor knowledge of the confining potential especially in what concerns its relativistic properties: the Lorentz structure (scalar, vector, etc.) and the dependence on the covariant variables such as $k^2=k_0^2-\mathbf{k}^2$. Nevertheless we can perform some general considerations and then apply them to a particular case of the linearly rising potential. To this end we note that for any nonrelativistic potential $V(-\mathbf{k}^2)$ the simplest relativistic generalization is to replace it by $V(k_0^2-\mathbf{k}^2)$.

In the case of the Lorentz-vector confining potential we can use the same approach as for the one-gluon exchange even with more general vertices containing the Pauli terms, since the mass-shell vector currents are conserved here as well. It is possible to introduce alongside with the "diagonal gauge" the so-called "instantaneous gauge" [9] which is the generalization of the Coulomb gauge:

$$V_V(k_0^2-\mathbf{k}^2)\bar u_a(\mathbf{p})\bar u_b(-\mathbf{p})\Gamma_a^\mu\Gamma_{b\mu}u_a(\mathbf{q})u_b(-\mathbf{q}) = \bar u_a(\mathbf{p})\bar u_b(-\mathbf{p})\Big\{V_V(-\mathbf{k}^2)\Gamma_a^0\Gamma_b^0 - \big[V_V(-\mathbf{k}^2)\mathbf{\Gamma}_a\cdot\mathbf{\Gamma}_b+V_V'(-\mathbf{k}^2)(\mathbf{\Gamma}_a\cdot\mathbf{k}) \times(\mathbf{\Gamma}_b\cdot\mathbf{k})\big]\Big\}u_a(\mathbf{q})u_b(-\mathbf{q}), \qquad (9)$$

where

$$V_V(k_0^2-\mathbf{k}^2)\cong V_V(-\mathbf{k}^2)+k_0^2V_V'(-\mathbf{k}^2)$$

and

$$k_0^2=(\epsilon_a(\mathbf{p})-\epsilon_a(\mathbf{q}))(\epsilon_b(\mathbf{q})-\epsilon_b(\mathbf{p}))\cong-\frac{(\mathbf{p}^2-\mathbf{q}^2)^2}{4m_am_b} \qquad (10)$$

with the correct Dirac limit in which the retardation contribution vanishes when one of the particles becomes infinitely heavy [10].

For the case of the Lorentz-scalar potential we can make the same expansion in k_0^2, which yields

$$V_S(k_0^2-\mathbf{k}^2)\cong V_S(-\mathbf{k}^2)+k_0^2V_S'(-\mathbf{k}^2). \tag{11}$$

But in this case we have no reasons to fix k_0^2 in the only way (10). The other possibility is to take a half sum instead of a symmetrized product, namely to set (see e. g. [11, 10])

$$\begin{aligned}k_0^2 &= \frac{1}{2}\left[(\epsilon_a(\mathbf{p})-\epsilon_a(\mathbf{q}))^2+(\epsilon_b(\mathbf{q})-\epsilon_b(\mathbf{p}))^2\right]\\ &\cong \frac{1}{8}(\mathbf{p}^2-\mathbf{q})^2\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right).\end{aligned} \tag{12}$$

The Dirac limit is not fulfilled by this choice, but this cannot serve as a decisive argument. Thus the most general expression for the energy transfer squared, which incorporates both possibilities (10) and (12) has the form

$$\begin{aligned}k_0^2 &= \lambda(\epsilon_a(\mathbf{p})-\epsilon_a(\mathbf{q}))(\epsilon_b(\mathbf{q})-\epsilon_b(\mathbf{p}))+(1-\lambda)\\ &\quad\times\frac{1}{2}\left[(\epsilon_a(\mathbf{p})-\epsilon_a(\mathbf{q}))^2+(\epsilon_b(\mathbf{q})-\epsilon_b(\mathbf{p}))^2\right]\\ &\cong -\lambda\frac{(\mathbf{p}^2-\mathbf{q}^2)^2}{4m_am_b}\\ &\quad+(1-\lambda)\frac{1}{8}(\mathbf{p}^2-\mathbf{q})^2\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right),\end{aligned} \tag{13}$$

where λ is the mixing parameter.

Thus the spin-independent part of $q\bar q$ potential with the account of retardation corrections takes the form:

$$\begin{aligned}V_{\rm SI}(r) &= V_C(r)+V_{\rm conf}(r)+V_{\rm VD}(r)\\ &+\frac{1}{8}\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right)\Delta\big[V_C(r)+(1+2\kappa)V_V(r)\big],\end{aligned} \tag{14}$$

where the velocity-dependent part

$$\begin{aligned}V_{\rm VD}(r) &= V_{\rm VD}^C(r)+V_{\rm VD}^V(r)+V_{\rm VD}^S(r), \qquad (15)\\ V_{\rm VD}^C(r) &= \frac{1}{2m_am_b}\left\{V_C(r)\left[\mathbf{p}^2+\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r^2}\right]\right\}_W\\ V_{\rm VD}^V(r) &= \frac{1}{m_am_b}\left\{V_V(r)\mathbf{p}^2\right\}_W\\ &+\frac{1}{4}\left[(1-\lambda_V)\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right)-\frac{2\lambda_V}{m_am_b}\right]\\ &\times\left\{V_V(r)\mathbf{p}^2+V_V'(r)\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r}\right\}_W,\\ V_{\rm VD}^S(r) &= \frac{1}{2}\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right)\left\{V_V(r)\mathbf{p}^2\right\}_W\\ &+\frac{1}{4}\left[(1-\lambda_S)\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right)-\frac{2\lambda_S}{m_am_b}\right]\\ &\times\left\{V_V(r)\mathbf{p}^2+V_V'(r)\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r}\right\}_W\end{aligned}$$

and $\{\dots\}_W$ denotes the Weyl ordering of operators. Making the natural decomposition

$$\begin{aligned}V_{\rm VD}(r) &= \frac{1}{m_am_b}\left\{\mathbf{p}^2V_{bc}(r)+\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r^2}V_c(r)\right\}_W\\ &+\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right)\left\{\mathbf{p}^2V_{de}(r)-\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r^2}V_e(r)\right\}_W\end{aligned} \tag{16}$$

we obtain [12] for the corresponding structures with $\lambda_V=1$ and including one-loop radiative corrections in $\overline{MS}$ renormalization scheme:

$$\begin{aligned}V_C(r) &= -\frac{4}{3}\frac{\bar\alpha_V(\mu^2)}{r}-\frac{4}{3}\frac{\beta_0\alpha_s^2(\mu^2)}{2\pi}\frac{\ln(\mu r)}{r},\\ V_{bc}(r) &= -\frac{2}{3}\frac{\bar\alpha_V(\mu^2)}{r}-\frac{2}{3}\frac{\beta_0\alpha_s^2(\mu^2)}{2\pi}\frac{\ln(\mu r)}{r}\\ &\quad+\left(\frac{1-\varepsilon}{2}-\frac{\varepsilon\lambda_S}{2}\right)Ar+B,\\ V_c(r) &= -\frac{2}{3}\frac{\bar\alpha_V(\mu^2)}{r}-\frac{2}{3}\frac{\beta_0\alpha_s^2(\mu^2)}{2\pi}\\ &\quad\times\left[\frac{\ln(\mu r)}{r}-\frac{1}{r}\right]-\left(\frac{1-\varepsilon}{2}+\frac{\varepsilon\lambda_S}{2}\right)Ar,\\ V_{de}(r) &= -\frac{\varepsilon}{4}(1+\lambda_S)Ar+B,\\ V_e(r) &= -\frac{\varepsilon}{4}(1-\lambda_S)Ar,\end{aligned} \tag{17}$$

where

$$\begin{aligned}\bar\alpha_V(\mu^2) &= \alpha_s(\mu^2)\left[1+\left(\frac{a_1}{4}+\frac{\gamma_E\beta_0}{2}\right)\frac{\alpha_s(\mu^2)}{\pi}\right],\\ a_1 &= \frac{31}{3}-\frac{10}{9}n_f, \quad \beta_0=11-\frac{2}{3}n_f.\end{aligned}$$

Here n_f is a number of flavours and μ is a renormalization scale. It is easy to check that the exact Barchielli, Brambilla, Prosperi relations [13] following from the Lorentz invariance of the Wilson loop

$$\begin{aligned}V_{de}-\frac{1}{2}V_{bc}+\frac{1}{4}(V_C+V_0)&=0,\\ V_e+\frac{1}{2}V_c+\frac{r}{4}\frac{{\rm d}(V_C+V_0)}{{\rm d}r}&=0\end{aligned} \tag{18}$$

are exactly satisfied.

The expression for spin-dependent part of the quark-antiquark potential with the inclusion of radiative corrections can be found in ref. [12]. Now we can calculate the mass spectra of heavy quarkonia with the account of all relativistic corrections (including retardation effects) of order v^2/c^2 and one-loop radiative corrections. For this

purpose we substitute the quasipotential which is a sum of the spin-independent and spin-dependent parts into the quasipotential equation. Then we multiply the resulting expression from the left by the quasipotential wave function of a bound state and integrate with respect to the relative momentum. Taking into account the accuracy of the calculations, we can use for the resulting matrix elements the wave functions of Eq. (1) with the static potential

$$V_{\rm NR}(r) = -\frac{4}{3}\frac{\bar\alpha_V(\mu^2)}{r} + Ar + B. \tag{19}$$

As a result we obtain the mass formula ($m_a = m_b = m$)

$$\frac{b^2(M)}{2\mu_R} = W + \langle a\rangle\langle \mathbf{L}\cdot\mathbf{S}\rangle + \langle b\rangle\Big\langle\Big[-(\mathbf{S}_a\cdot\mathbf{S}_b) + \frac{3}{r^2}(\mathbf{S}_a\cdot\mathbf{r})(\mathbf{S}_b\cdot\mathbf{r})\Big]\Big\rangle + \langle c\rangle\langle\mathbf{S}_a\cdot\mathbf{S}_b\rangle, \tag{20}$$

where the first term on the right-hand side of the mass formula contains all spin-independent contributions, the second term describes the spin-orbit interaction, the third term is responsible for the tensor interaction, while the last term gives the spin-spin interaction.

To proceed further we need to discuss the parameters of our model. There is the following set of parameters: the quark masses (m_b and m_c), the QCD constant Λ and renormalization point μ in the short-range part of the $Q\bar Q$ potential, the slope A and intercept B of the linear confining potential (8), the mixing coefficient ε (7), the long-range anomalous chromomagnetic moment κ of the quark (6), and the mixing parameter λ_S in the retardation correction for the scalar confining potential. We can fix the values of the parameters $\varepsilon = -1$ and $\kappa = -1$ from the consideration of radiative decays [3] and comparison of the heavy quark expansion in our model [14, 15] with the predictions of the heavy quark effective theory. We fix the slope of the linear confining potential $A = 0.18$ GeV^2 which is a rather adopted value. In order to reduce the number of independent parameters we assume that the renormalization scale μ in the strong coupling constant $\alpha_s(\mu^2)$ is equal to the quark mass. We also varied the quark masses in a reasonable range for the constituent quark masses.

State	Particle	Theory	Experiment [18]
1^1S_0	η_c	2.979	2.9798
1^3S_1	J/Ψ	3.096	3.09688
1^3P_0	χ_{c0}	3.424	3.4173
1^3P_1	χ_{c1}	3.510	3.51053
1^3P_2	χ_{c2}	3.556	3.55617
2^1S_0	η_c'	3.583	3.594
2^3S_1	Ψ'	3.686	3.686
1^3D_1		3.798	3.7699
1^3D_2		3.813	
1^3D_3		3.815	
2^3P_0	χ_{c0}'	3.854	
2^3P_1	χ_{c1}'	3.929	
2^3P_2	χ_{c2}'	3.972	
3^1S_0	η_c''	3.991	
3^3S_1	Ψ''	4.088	4.040
2^3D_1		4.194	4.159
2^3D_2		4.215	
2^3D_3		4.223	

Table 1: Charmonium mass spectrum.

The numerical analysis and comparison with experimental data lead to the following values of our model parameters: $m_c = 1.55$ GeV, $m_b = 4.88$ GeV, $A = 0.18$ GeV^2, $B = -0.16$ GeV, $\mu = m_Q$, $\Lambda = 0.178$ GeV, $\varepsilon = -1$, $\kappa = -1$, $\lambda_S = 0$. The quark masses $m_{c,b}$ have usual values for constituent quark models and coincide with those chosen in our previous analysis [8]. The above value of the retardation parameter λ_S for the scalar confining potential coincides with the minimal area low and flux tube models [16], with lattice results [17] and Gromes suggestion [11]. The found value for the QCD parameter Λ gives the following values for the strong coupling constants $\alpha_s(m_c^2) \approx 0.32$ and $\alpha_s(m_b^2) \approx 0.22$.

The results of our numerical calculations of the mass spectra of charmonium and bottomonium (in GeV) are presented in Tables 1 and 2. We see that the calculated masses agree with experimental values within few MeV and this difference is compatible with the estimates of the higher order corrections in v^2/c^2 and α_s. The model reproduces correctly both the positions of the centres of gravity of the levels and their fine and hyperfine splitting. Note that the good agreement of the calculated mass spectra with experimental data is achieved by systematic ac-

State	Particle	Theory	Experiment [18]
1^1S_0	η_b	9.400	
1^3S_1	Υ	9.460	9.46037
1^3P_0	χ_{b0}	9.864	9.8598
1^3P_1	χ_{b1}	9.892	9.8919
1^3P_2	χ_{b2}	9.912	9.9132
2^1S_0	η_b'	9.990	
2^3S_1	Υ'	10.020	10.023
1^3D_1		10.151	
1^3D_2		10.157	
1^3D_3		10.160	
2^3P_0	χ_{b0}'	10.232	10.232
2^3P_1	χ_{b1}'	10.253	10.2552
2^3P_2	χ_{b2}'	10.267	10.2685
3^1S_0	η_b''	10.328	
3^3S_1	Υ''	10.355	10.3553
2^3D_1		10.441	
2^3D_2		10.446	
2^3D_3		10.450	
3^3P_0	χ_{b0}''	10.498	
3^3P_1	χ_{b1}''	10.516	
3^3P_2	χ_{b2}''	10.529	
4^1S_0	η_b'''	10.578	
4^3S_1	Υ'''	10.604	10.580

Table 2: Bottomonium mass spectrum.

counting for all relativistic corrections (including retardation corrections) of order v^2/c^2, both spin-dependent and spin-independent ones, while in most of potential models only the spin-dependent corrections are included.

The calculated mass spectra of charmonium and bottomonium are close to the results of our previous calculation [8] where retardation effects in the confining potential and radiative corrections to the one-gluon exchange potential were not taken into account. Both calculations give close values for the experimentally measured states as well as for the yet unobserved ones. The inclusion of radiative corrections allowed to get better results for the fine splittings of quarkonium states. Thus we can conclude from this comparison that the inclusion of retardation effects and spin-independent one-loop radiative corrections resulted only in the slight shift ($\approx 10\%$) in the value of the QCD parameter Λ and an approximately two-fold decrease of the constant B. Such changes of parameters almost do not influence the wave functions. As a result the decay matrix elements involving heavy quarkonium states remain mostly unchanged.

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